Mechanics Corner

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Braced Cantilever

1 Introduction

Anyone who has actually gotten into machine design is familiar with this difficulty. Consider the situation where a project is well advanced, many plans have been made, and it is all based on the assumption of the adequacy of one particular part. When you finally get to the detailed analysis of that part, the calculations show that it is not adequate. What can you do?

To make the problem much more concrete, consider the cantilever beam shown in Figure 1. It supports a weight W at the free end, and when someone finally makes the calculation, the tip deflection, δ, is unacceptably large. The whole system design has been developed on the assumed adequacy of that cantilever, and there is no room to put in a beam with a larger section to give more stiffness. What can be done?

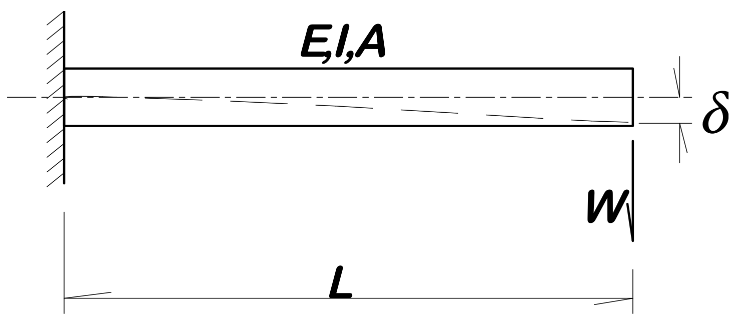

Figure 1: Unstayed Beam With Load

Any of countless machine design texts, mechanics of materials texts, etc., give the formula for the end deflection,

$$\delta = \frac{WL^3}{3EI} \tag{1}$$

where

E = Young's modulus for the beam material

I = area moment of inertia for the beam cross section

L = length of the beam

W = tip load value

While we can argue that someone should have checked this earlier, finger-pointing does not fix the problem.

2 Add a Tie-Rod Backstay

Consider the addition of a tie-rod backstay as shown in Figure 2. The rod is relatively slender, and is pin jointed at both ends; it carries tension only.

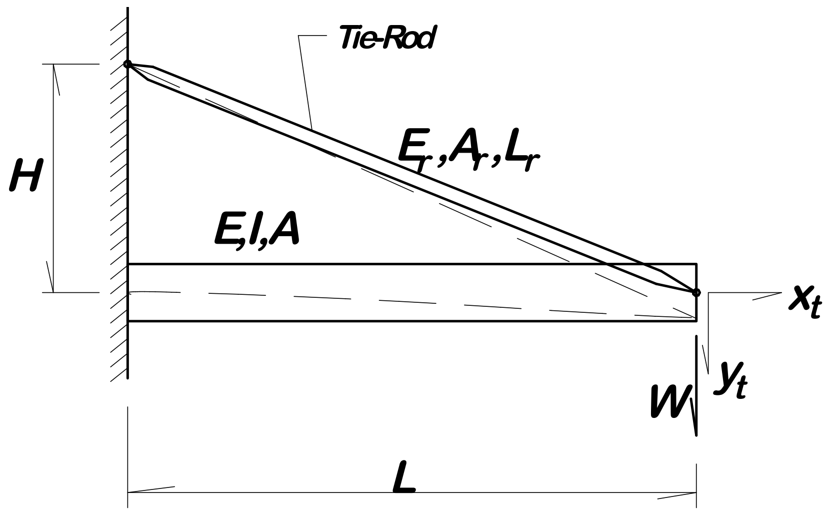

Figure 2: Cantilever Beam With Cable Backstay

As before, E, I, A with no subscripts represent the Young's modulus, area moment of inertia, and cross sectional area of the cantilever beam. The new quantities involved are

E_r = effective Young's modulus for the tie-rod;

A_r = effective cross sectional area for the tie-rod;

L_r = unstressed length of the tie-rod between the connection points;

x_t = assumed axial displacement of the right end;

y_t = assumed downward displacement of the right end.

The difficulty at hand is that now the vertical load W is shared between the tie-rod tension (T) and shear and bending in the beam.

2.1 Tie-Rod Tension

First, let us consider the matter of stretch in the rod, assuming the ends have the assumed displacements x_t and y_t as indicated in Figure 2. The length of the stretched rod is L_c^*,

$$
\begin{aligned}
(L_r^*)^2 &= (L + x_t)^2 + (H + y_t)^2 \\
&= +L^2 + H^2 + 2Hy_t + 2Lx_t + x_t^2 + y_t^2 \\
&= L_r^2 + 2Hy_t + 2Lx_t + x_t^2 + y_t^2 \qquad (2)
\end{aligned}
$$

The strain in the tie-rod is then

$$
\begin{aligned}
\varepsilon_r &= \frac{L_r^* - L_r}{L_r} \\
&= \frac{L_r^* - L_r}{2L_r} \cdot \frac{2L_r}{L_r} \\
&\approx \frac{(L_r^* - L_r)(L_r^* + L_r)}{2L_r^2} \\
&= \frac{\left(L_r^{*^2} - L_r^2\right)}{2L_r^2} \\
&= \frac{2Hy_t + 2Lx_t + x_t^2 + y_t^2}{2L_r^2} \\
&\approx \frac{x_t}{L_r} \cdot \frac{L}{L_r} + \frac{y_t}{L_r} \cdot \frac{H}{L_r} \\
&= \frac{x_t}{L_r} \cdot \cos\phi + \frac{y_t}{L_r} \cdot \sin\phi \qquad (3)
\end{aligned}
$$

where ϕ = angle between tie-rod and the beam undeformed centerline (note that $2L_r \approx L_r + L_r^*$). This expression is correct to the first order in the small quantities x_t/L and y_t/L. The tie-rod tension is then

$$
T = A_r E_r \varepsilon_r = A_r E_r \left(\frac{x_t}{L_r} \cdot \cos\phi + \frac{y_t}{L_r} \cdot \sin\phi \right) \qquad (4)
$$

2.2 Beam Loads

Assume that the right end of the beam is subject to a downward vertical load F_y and a horizontal load F_x to the right on the beam. The we know that the tip loads and displacements of the beam will be such that

$$
\begin{aligned}
F_x &= AE\frac{x_t}{L} \qquad (5) \\
F_y &= \frac{3EI}{L^3} y_t \qquad (6)
\end{aligned}
$$

2.3 Equilibrium Condition

The condition is for equilibrium at the junction point of the beam, the tie-rod, and the applied load W. The condition for equilibrium is

$$\sum^{+\rightarrow} F_{Horiz} = -F_x - T\cos\phi = 0 \quad (7)$$

$$\sum^{+\uparrow} F_{Vert} = F_y + T\sin\phi - W = 0 \quad (8)$$

At this point, we must substitute for the forces in terms of the disagreements, to obtain

$$\begin{aligned} -AE\frac{x_t}{L} - \left[A_rE_r\left(\frac{x_t}{L_r}\cdot\cos\phi + \frac{y_t}{L_r}\cdot\sin\phi\right)\right]\cos\phi &= 0 \\ 3\frac{EI}{L^3}y_t + \left[A_rE_r\left(\frac{x_t}{L_r}\cdot\cos\phi + \frac{y_t}{L_r}\cdot\sin\phi\right)\right]\sin\phi &= W \end{aligned}$$

or

$$\begin{bmatrix} -\frac{AE}{L} - \frac{A_rE_r}{L_r}\cos\phi & -\frac{A_cE_c}{L_c}\sin\phi \\ \frac{A_rE_r}{L_r}\cos\phi & \frac{3EI}{L^3} + \frac{A_rE_r}{L_r}\sin\phi \end{bmatrix} \begin{Bmatrix} x_t \\ y_t \end{Bmatrix} = \begin{Bmatrix} 0 \\ W \end{Bmatrix} \quad (9)$$

At this point, it is a matter of solving numerically for the displacements, x_t and y_t and then substitute back to obtain the forces.

2.4 Numerical Calculation

To make the above more specific, suppose that the following numerical values apply.

The applied load (for all cases) is

$W = 31400$ N

For the beam alone —

$L = 2.25$ m

$AE = 5.101538 \cdot 10^9$ N

$EI = 973607$ N/m^2

With these values, eq(1) gives a tip deflection of

$\delta = 0.122454$ m or 122 mm.

For the tie-rod alone —

$A_rE_r = 0.831317 \cdot 10^8$ N

$L_r = 2.304886$ m

With the load supported by the combination of the cantilever beam and the tie-rod, the results are

$x_t = -0.0005308$ m

$y_t = 0.0181471$ m

$T = 123295.97$ N

$F_x = -120359.929$ N

$F_y = 4653.349$ N

These numerical values show that the addition of the tie-rod brace substantially reduces the tip vertical deflection (from 122.4 mm to 18.1 mm). This is accomplished at the price of inducing moderately heavy loads on the supporting wall, 123296 N at the tie-rod and 120360 N at the supported end of the beam.

3 Pin Connected Alternative Frame

The numbers developed in the preceding numerical calculations show that, with the tie-rod backstay in place, there is relatively little bending in the beam, so the moment carrying capacity of the beam is no longer of major importance (it was all important when there was no brace at all!). This suggests exploring the alternative of using a pin-connected frame in place of the beam and tie-rod arrangement. Such a pin-connected frame is shown in Figure 3.

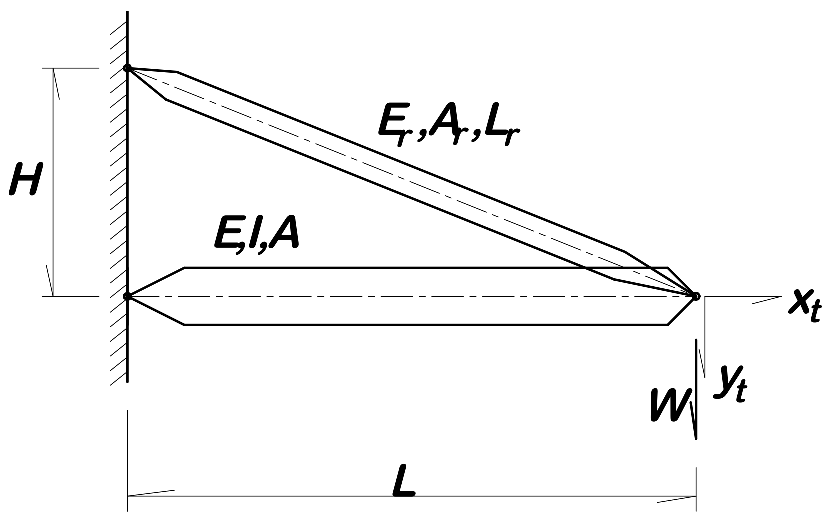

Figure 3: Pin-Connected Frame

Note particularly that there are pin connections at the left end of the beam, the left end of the brace (now called a rod), and a pin connection where the two join at the right and support the load W. The beam properties remain E, I, and A, but the end conditions are now completely altered. For the brace, there was no moment at either end previously, and that condition applies here as well. Each member is now only a two-force member, meaning then that the forces act along the member axes.

3.1 Brace Tension Force

The free length of the brace is $L_r = \left(L^2 + H^2\right)^{1/2}$, and the length under load is $L_r^* = A_r E_r \varepsilon_r$ and all of the analysis above relating the tension to the displacements x_t and y_t applies again in this case:

$$T = A_r E_r \left(\frac{x_t}{L_r} \cdot \cos\phi + \frac{y_t}{L_r} \cdot \sin\phi \right) \tag{10}$$

3.2 Beam Axial Force

As before, we assume that the beam is subject to an axis tensile load F_x causing the tip displacement x_t:

$$F_x = AE \frac{x_t}{L} \tag{11}$$

3.3 Equilibrium

The equilibrium of the pin connecting the two members and the load is described by

$$\sum^{+\rightarrow} F_{Horiz} = -F_x - T\cos\phi = 0 \tag{12}$$

$$\sum^{+CCW} M_{Origin} = -LW + LT\sin\phi = 0 \tag{13}$$

With the solutions

$$T = \frac{W}{\sin\phi} \tag{14}$$

$$F_x = -W\cot\phi \tag{15}$$

which give

$$x_t = \frac{-LW\cot\phi}{AE} \tag{16}$$

$$y_t = W\left(\frac{L_r}{A_r E_r \sin^2\phi} + \frac{L}{AE} \right) \tag{17}$$

$$\begin{aligned} T &= A_r E_r \left(\frac{x_t}{L_r} \cdot \cos\phi + \frac{y_t}{L_r} \cdot \sin\phi \right) \\ y_t &= W\left(\frac{L_r}{A_r E_r \sin^2\phi} + \frac{L}{AE} \right) \end{aligned} \tag{18}$$

3.4 More Numbers

For the entirely pin-connected frame, the member values will be taken the same as previously. All that is different is that there is now to be no moment in the beam at the built-in end.

The previous results and those for this case are summarized in the following table for comparison.

	Beam Alone	Beam With Tie-Rod	Pin-Connected Frame	
$x_t =$	˜	-0.5308	-0.6232	(mm)
$y_t =$	122.45	18.147	21.304	(mm)
$T =$	˜	123296	144747	(N)
$F_x =$	0	-120360	-141300	(N)
$F_y =$	31400	4653.3	0.0	(N)

4 Conclusion

Looking back at this problem from a sort of first-last-middle perspective, it is evident that:

1. The built–in beam carries the load in a mix of shear and bending, but the resulting tip deflection is quite large;

2. The pin-jointed frame carries the load entirely as member tensions (there are no shear or bending moment in any member) with rather small tip displacement but at the expense of fairly large axial loads which go into the supports;

3. Imposing a greater degree of internal constraint by building in the end of the beam while maintaining the tie-rod backstay reduces the axial loads somewhat, but at the expense of creating an end moment on the support.

There is no answer to the question "Which is the best solution?" until we know more about what the supporting wall can best withstand. The one thing that was clear from the beginning is that the built-in beam alone was not adequate from the perspective of tip deflection. It is also evident that we do have some options.

DrD is a retired Professor of Mechanical Engineering in the USA. He can be reached for comments, questions, or requests at through the ME Forum. Be sure to check back soon at www.http://mechanical-engineering.in/forum/blog/206-mechanics-corner/ for more articles.