Mechanics Corner

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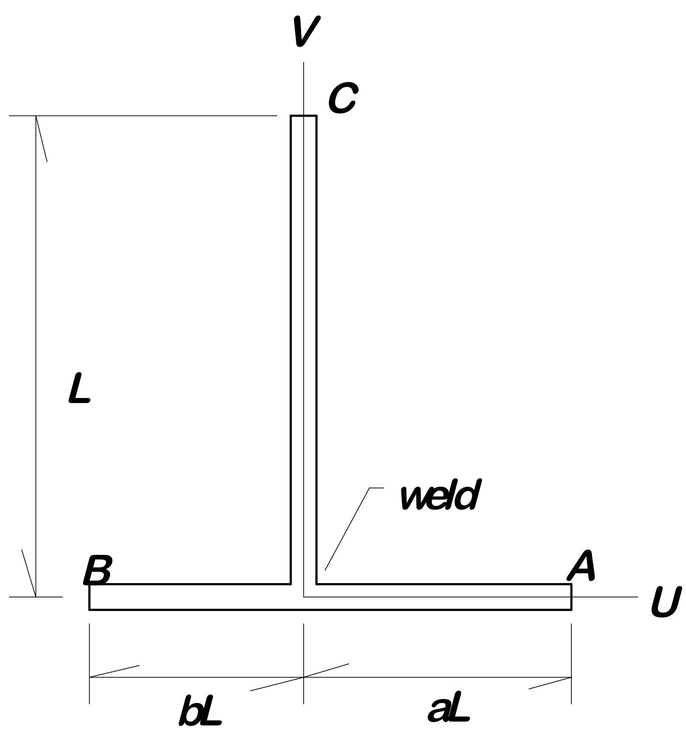
A Problem in Statics & Dynamics

1 Introduction

A problem was recently posted on this Forum, requesting help, that has led me to consider a somewhat more general problem for this post. The scope of this post will include the original problem, although not by the method required there, but will also go beyond to a more general geometry. We begin here by stating the present problem; interested readers are invited to search back for the original problem posted 19 December, 2016, by *iivii*.



Assembly Drawing, with Dimensions

2 Present Problem

Assume that uniform rod stock is available with a linear density of $\mu =$ mass/length. Two pieces are cut, lengths L and $(a+b)\,L$, where $L > 0$, $a \geq 0$, and $b \geq 0$. The two rods are welded together to form the component shown in the figure above. Assume that the assembly is supported from a pin at point C in a vertical gravitational field. What is the static angular position? If the entire assembly is allowed to swing as a pendulum, what is the natural frequency for small amplitudes?

3 Solution

For a problem like this, there are several questions we should probably seek to answer first before we move to the actual questions asked in the problem statement. These introductory questions include:

1. What is the total mass of the assembly?
2. Where is the center of mass?
3. What is the mass moment of inertia of the assembly with respect to point C?

If we can answer all of these preliminaries, then the actual questions asked in the problem statement should not be too difficult. Without these fundamentals, most questions of significance cannot be answered.

3.1 Total Mass

The first of the preliminaries is the matter of the total mass. We know the mass per unit length, μ, so this simply requires that we find the total length and multiply. Thus consider

$M = \mu aL + \mu bL + \mu L = \mu L\left(1 + a + b\right)$ — total mass

3.2 Center of Mass Location

It is convenient to locate the center of mass in the body coordinate system $U - V$, to be denoted as (u_c, v_c). The required relations are

$$Mu_c = \frac{aL}{2}\left(\mu aL\right) + \left(\frac{-bL}{2}\right)\left(\mu bL\right) + 0\left(\mu L\right) = \frac{\mu L^2}{2}\left(a^2 - b^2\right)$$

$$Mv_c = 0\left(\mu aL\right) + (0)\left(\mu bL\right) + \frac{L}{2}\left(\mu L\right) = \frac{\mu L^2}{2}$$

Using the previous expression for M gives

$u_c = \dfrac{\frac{\mu L^2}{2}\left(a^2-b^2\right)}{\mu L(1+a+b)} = \dfrac{L\left(a^2-b^2\right)}{2(a+b+1)}$ — horizontal CM location

$v_c = \dfrac{\frac{\mu L^2}{2}}{\mu L(1+a+b)} = \dfrac{L}{2(a+b+1)}$ — vertical CM location

3.3 Mass Moment of Inertia with Respect to Point C

The entire assembly can be through of as comprised of three rods, having lengths aL, bL, and L. The center of mass of each rod individually is easily located at the center. Then, we need two concepts to determine the mass moment of inertia with respect to C:

(a) The MMOI of a straight rod, with respect to its own center of mass is $J_{ic} = \frac{1}{12} m_i l_i^2$ where m_i is the rod mass and l_i is the rod length;

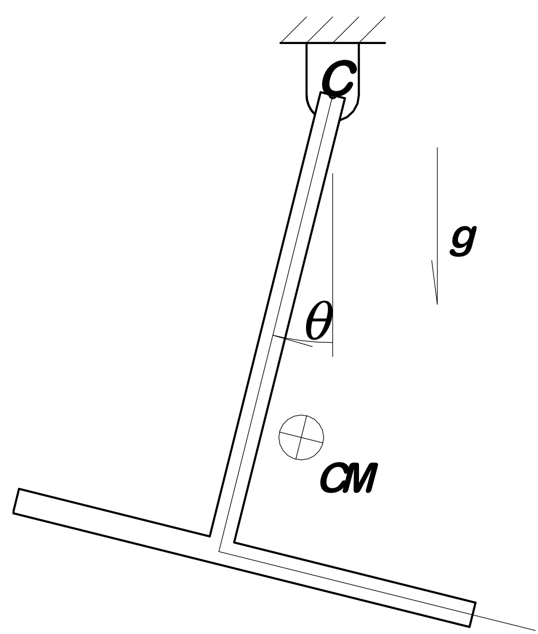
(b) The parallel axis theorem tells us that the moment of inertia about a point C other than the body center of mass is given by $J_C = J_{ic} + mD^2$ where J_{ic} is the MMOI with respect to the body center of mass, and D is the distance from the body center of mass to point C.

Applying these two ideas to the problem at hand gives

$$
\begin{aligned}
J_C = &\frac{1}{12}(\mu aL)(aL)^2 + (\mu aL)\left[\left(\frac{aL}{2}\right)^2 + L^2\right] \\
&+\frac{1}{12}(\mu bL)(bL)^2 + (\mu bL)\left[\left(\frac{bL}{2}\right)^2 + L^2\right] \\
&+ \frac{1}{12}(\mu L)(L^2) + (\mu L)\left(\frac{L}{2}\right)^2 \\
&= \frac{1}{3}L^3\mu\left(a^3 + 3a + b^3 + 3b + 1\right) \qquad \text{— MMOI for composite body wrt } C
\end{aligned}
$$

4 Statics Problem

Next, consider the assembly suspended by a pin at point C as shown below:



Hanging Body

It is evident that, for the dimensions shown in the drawing, it will hang about as shown, but how do we determine the exact equilibrium value of angle θ? The Principle of Virtual Work will be employed, with the work represented by a potential energy expression.

Taking zero gravitational potential energy at the level of the pin C, the potential energy in the position shown is

$V = -Mg\left[(L - v_c)\cos\theta + u_c \sin\theta\right]$

where

$M = \mu L\,(1 + a + b) =$ total mass of the assembly (as determined above);

$g =$ acceleration of gravity;

$\theta =$ inclination angle

Substituting the expressions found previously for u_c and v_c gives

$$
\begin{aligned}
V(\theta) &= -Mg\left\{\left[L - \tfrac{L}{2(a+b+1)}\right]\cos\theta + \left[\tfrac{L\left(a^2-b^2\right)}{2(a+b+1)}\right]\sin\theta\right\} \\
&= -\mu L\,(1+a+b)\,g\left\{\left[L - \tfrac{L}{2(a+b+1)}\right]\cos\theta + \left[\tfrac{L\left(a^2-b^2\right)}{2(a+b+1)}\right]\sin\theta\right\} \\
&= -\tfrac{1}{2}L^2 g\mu\left[(\sin\theta)\,a^2 + 2\,(\cos\theta)\,a - (\sin\theta)\,b^2 + 2\,(\cos\theta)\,b + \cos\theta\right] \\
&= -\tfrac{1}{2}L^2 g\mu\left[(a^2-b^2)\sin\theta + (2a+2b+1)\cos\theta\right]
\end{aligned}
$$

The expression above gives the potential energy of the assembly at any angular position θ. At the equilibrium position, $\theta = \theta_{Eq}$, this expression is a minimum. Thus a differentiation is required,

$\frac{dV(\theta)}{d\theta} = \frac{\mu g L^2}{2}\left[(2a+2b+1)\sin\theta + (b^2-a^2)\cos\theta\right]$

Setting this equal to zero and solving for the angle gives

$\theta_{Eq} = \arctan\left(\frac{a^2-b^2}{2a+2b+1}\right)$ — equilibrium angle

As a check on this result, consider what intuition says about this matter. It is intuitively evident that, at equilibrium, the center of mass should be directly below the suspension point. This requires that

$\theta_{Eq} = \arctan\left(\frac{u_c}{L-v_c}\right)$

On further substitution in this last expression, it is found to be identical to that obtained previously.

5 Dynamics Problem

The dynamics problem given was to determine the natural frequency of small amplitude vibrations when the suspended assembly swings as a pendulum. This requires that we first obtain the equation of motion for finite amplitudes, secondly that we linearize this equation of motion for small amplitudes, and finally that we extract the natural frequency.

With the potential energy already in hand (above), it is most natural to use an energy approach to obtain the equation of finite amplitude motions. Thus we begin by formulating the kinetic energy of the swinging system.

$T = \frac{1}{2} J_C \dot{\theta}^2$

where J_C = MMOI wrt the pivot point C, already determined above.

Since J_C is a constant value, it has zero derivative and therefore the centripetal coefficient is zero (see post #11 on Eksergian's Equation of Motion for the definition of the centripetal coefficient). The effective inertia is simply J_C, so the equation of motion is

$$J_C \ddot{\theta} + \frac{dV}{d\theta} = \frac{1}{3} L^3 \mu \left(a^3 + 3a + b^3 + 3b + 1\right) \ddot{\theta}$$
$$+ \frac{\mu g L^2}{2} \left[(2a + 2b + 1) \sin\theta + \left(b^2 - a^2\right) \cos\theta\right] = 0$$

— equation of motion

Now, for small angular motions about equilibrium, let $\theta = \theta_{Eq} + \phi$, where ϕ is a small displacement from the equilibrium position. The equation of motion becomes

$$\frac{1}{3} L^3 \mu \left(a^3 + 3a + b^3 + 3b + 1\right) \ddot{\phi}$$
$$+ \frac{\mu g L^2}{2} \left[(2a + 2b + 1) \sin\left(\theta_{Eq} + \phi\right) + \left(b^2 - a^2\right) \cos\left(\theta_{Eq} + \phi\right)\right] = 0$$

When the sine and cosine sums are expanded, they are approximately

$$\sin\left(\theta_{Eq} + \phi\right) \approx \sin\theta_{Eq} + \phi \cos\theta_{Eq}$$
$$\cos\left(\theta_{Eq} + \phi\right) \approx \cos\theta_{Eq} - \phi \sin\theta_{Eq}$$

When these approximations are substituted into the equation of motion, the result is

$$\frac{1}{3} L^3 \mu \left(a^3 + 3a + b^3 + 3b + 1\right) \ddot{\phi}$$
$$+ \frac{\mu g L^2}{2} \left[(2a + 2b + 1) \left(\sin\theta_{Eq} + \phi \cos\theta_{Eq}\right) + \left(b^2 - a^2\right) \left(\cos\theta_{Eq} - \phi \sin\theta_{Eq}\right)\right] = 0$$

— linearized equation of motion

Two terms on the left cancel due to the equilibrium condition, leaving

$$\frac{1}{3} L^3 \mu \left(a^3 + 3a + b^3 + 3b + 1\right) \ddot{\phi} + \frac{\mu g L^2}{2} \phi \left[(2a + 2b + 1) \cos\theta_{Eq} + \left(a^2 - b^2\right) \sin\theta_{Eq}\right] = 0$$

This is the linearized equation of motion governing small oscillations about equilibrium. The natural frequency is extracted from the coefficients thus:

$$\omega_n = \left\{ \frac{\frac{\mu g L^2}{2} \left[(2a+2b+1) \cos\theta_{Eq} + \left(a^2 - b^2\right) \sin\theta_{Eq}\right]}{\frac{1}{3} L^3 \mu \left(a^3+3a+b^3+3b+1\right)} \right\}^{1/2}$$

— undamped natural frequency

This answers the last question of the problem statement, the expression for the natural frequency of small oscillations about equilibrium. It is interesting to note the way the length factors a and b enter into this expression.

6 Numerical Cases

Two numerical cases are considered for examples. The first is a numerical form for the case proposed by *iivii* in the original Forum question where $a = b = 1/4$, and the second will be similar but with $b = 0$. For numerical purposes, assume that

$L = 0.85$ m

$\mu = 0.35$ kg/m

$g = 9.807$ m/s^2

6.1 Original Problem, $a = b = 1/4$

For the assumed numerical values, and with $a = b = 0.25$, the equilibrium angle is found to be zero, as expected (the assembly is symmetric). The oscillatory natural frequency is

$\omega_n = 3.69787$ rad/s

6.2 Asymmetrical Case, $a = 1/4,\ b = 0$

For the asymmetrical case ($b = 0$), the equilibrium angle is

$\theta_{Eq} = 2.3859^o$

and the natural frequency is

$\omega_n = 3.83609$ rad/s

It is not surprising that the second example gives a higher natural frequency. There are two effects at work to produce that result. The first is simply that there is less mass in the assembly (the B leg has been removed). Secondly, the small nonzero equilibrium angle tends to reduce the stiffness very slightly.

7 Conclusion

This note demonstrates several points. They include —

1. The process of finding an equilibrium position by energy methods;

2. The process of writing the exact equation of motion for a physical pendulum;

3. The process for linearizing the pendulum equation of motion about the equilibrium position;

4. The considerable complication introduced when a relatively simple problem is to be worked in general terms (leg lengths aL and bL, rather than specific values);

5. The complications introduced by asymmetry.

All of these are points worthy of note. The generalized problem has value in certain circumstances, such as to avoid having to work a dozen similar cases, one by one, but it also is considerably more complex. As engineers, we must choose wisely to work effectively.