Mechanics Corner

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Modeling Hysteresis

1 Introduction

What do you know about hysteresis? Many Mechanical Engineers will associate this term with the magnetization curve of a piece of magnetic material, and quickly conclude, "I don't have to worry about that!" But that would be wrong. While hysteresis does occur in magnetic systems, it happens in many other situations as well, many of them situations of concern to mechanical engineers.

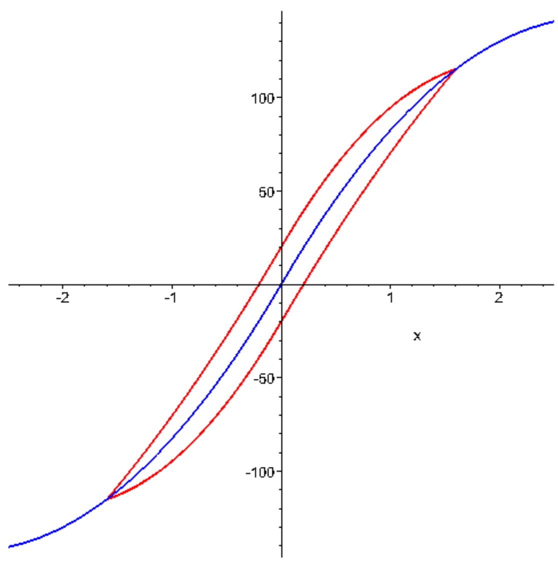

Figure 1: Typical Hysteresis Curve

Figure 1 shows a typical hysteresis curve, and it makes no difference as to what physical phenomena are involved. The red curve is the actual hysteresis curve. The blue curve is called the "spine."

In order to be more concrete in the discussion, suppose that this is the hysteresis curve for a mechanical spring. The deformation of the spring is the variable x, and the $y-$axis shows the force transferred through the spring. Imagine that you start with the undeformed spring, that is, $x = 0$. How much force do you expect to have in the spring? With no deformation, zero force is expected, so that operating condition of the spring is at the origin. Now, imagine that you load the spring infinitely slowly (the emphasis on the *infinitely* part!) For this infinitely slow process, the operating point is expected to move up the blue curve. The blue curve is curved, indicating that the spring is not quite linear; it is a *softening spring*. (To be a softening spring means that the stiffness decreases with increased deformation, just as indicated by the blue curve.)

But now you want to finish this experiment in this lifetime, so you push the spring a little bit faster. What happens? The operating point suddenly shifts to the upper red curve. It takes more force to get the same increase in deformation. The operating point is on the upper curve because that is the *loading curve*. If you now stop the loading process, the operating point returns to the blue curve, just as at the upper right end of the loop. Next, begin to unload the spring, that is, to gradually release it. Where is the operating point? Is is on the *unloading curve*, the lower red curve.

Suppose next that you begin with the spring compressed, allow it to unload all the way to the bottom left corner (this requires that you actually pull the spring, that is, apply a negative force.) From that point, the spring is again loaded back to the condition at the upper right corner. In this process, the operating point traces out the entire red curve. But notice that moving to the left is on the lower side of the red loop, while moving to the right is on the upper side of the loop.

Question: When the system is restored to the original operating point (the upper right corner), will there have been any net work done on the system? Everybody knows that springs are usually conservative, so the answer seems to be "no." But this spring is **not conservative**. Tracing out the loop requires work input to the spring, work proportional to the area of the loop (sounds like IC engine thermo, doesn't it?). That work input is required because of hysteretic losses within the material.

All real materials exhibit some degree of hysteresis, so this is not an unimportant topic. Some materials show a lot of hysteresis, such as rubber-type materials. Others, like steel, have only a slight amount of hysteresis. But the key thing to remember is that all materials have some, so the user must decide whether hysteresis is an important part of a particular problem or not.

2 Modeling Hysteresis

2.1 Preliminaries: The Signum Function

Mechanical hysteresis is an energy loss mechanism, much like dry friction or viscous friction. The last two are much more familiar, primarily because they are mathematically less complicated. Viscous friction can be included in an equation of motion by simply adding another term, a term of the form $b\dot{x}$. Dry friction, also called Coulomb friction, occurs in dry rubbing contacts where the friction force is proportional to the normal contact force. To include dry friction in an equation of motion usually involves add a term of the form $\mu N \cdot sgn(\dot{x})$. The last factor, $sgn(\dot{x})$ (pronounced signum x-dot) involves the use of the signum function defined as

$$sgn(x) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{1}$$

The purpose of the signum function in the expression for dry friction is to make the sign reverse when the direction of relative motion reverses. It will be similarly useful in modeling hysteresis.

2.2 The Spine

The blue curve shown in Figure 1 is the spine. It represents the "centerline" of the hysteresis loop (recognizing that it is a curve, not a simple line). The shape of the spine has a major impact on the overall shape of the hysteresis loop, so it is the logical place to begin.

Near the origin, the spine is nearly a straight line, but well away from the origin it departs from that initial straight line. This suggests that the spine be modeled by the function $S(x)$

$$S(x) = K\left(1 + \alpha |x|^{n-1}\right) x \tag{2}$$

For positive values of x, this gives a deviation from the straight line proportional to x^n, where n can be chosen at will. For $n = 2$, the deviation is proportional to x^2; for $n = 3$, the deviation is proportional to x^3, etc. By leaving one factor x outside the absolute value, the sign of the term is such as to make for an antisymmetric curve, typical of physical processes. If $\alpha > 0$, the spine will turn up at the right end. For the curve shown in Figure 1, $\alpha = -0.175$, which causes the tip to bend down on the right end. Also, in Figure 1, $n = 2$ is used. The analyst has complete freedom to choose these parameters in order to best fit the observed physical characteristics of the system at hand.

2.3 Direction Shift Term

The direction shift term is the term that reflects the effect of x increasing versus x increasing. As mentioned about this is easily handled with a factor $sgn(\dot{x})$ applied to a second factor that is always nonnegative. That second factor show go to zero at both extreme positions, so that the curve will come back to the spine. The direction shift term choice made for Figure 1 is

$$D(x, \dot{x}) = 20\, sgn(\dot{x}) \left(1 - \frac{x^2}{x_m^2}\right) \tag{3}$$

where the position is varying according to

$$x(t) = x_m \cos\left(\frac{2\pi t}{\tau}\right) \tag{4}$$

with

$x_m = 1.6$

$\tau = 2.2$

The important point is that this becomes

$$D(x, \dot{x}) = 32\, sgn(\dot{x}) \sin^2\left(\frac{2\pi t}{\tau}\right) \tag{5}$$

and the sine square goes to zero every half period.

There are certainly other ways to construct this function; this is shown simply as an example of one way that works.

2.4 Complete Expression

The complete expression for the force through the spring is simply the sum of the two terms, the spine and the direction shift term. Thus,

$$F\left(x,\dot{x}\right)=100\left(1-0.175\left|x\right|\right)x+32\,sgn\left(\dot{x}\right)\,\sin^{2}\left(\frac{2\pi t}{\tau}\right) \tag{6}$$

for this example.

3 Closure

There is great flexibility in the choice of constants, and the interested reader is encouraged to experiment with these functional forms, trying various constants to see what their effect is. This relatively simple process makes it possible to include hysteresis in system mathematical descriptions of all sorts, so there is really no justification for avoiding this issue.

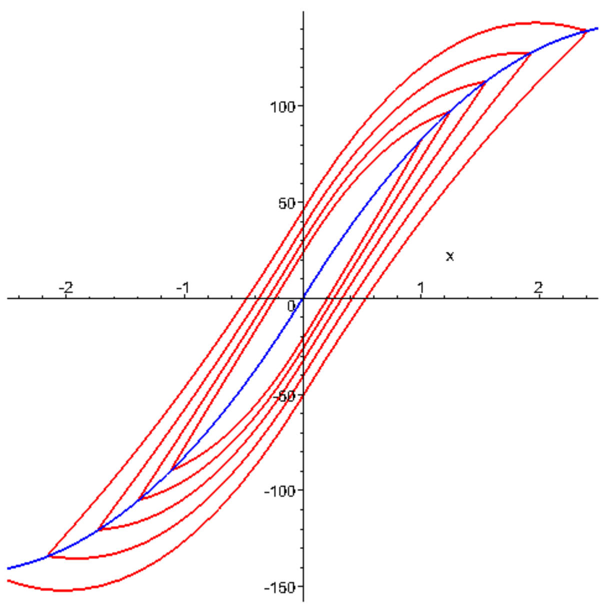

Figure 2: Another Hysteresis Example

DrD is a retired Professor of Mechanical Engineering in the USA. He can be reached for comments, questions, or requests through the ME Forum message system Be sure to check back soon at Mechanics Corner for more articles.